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Gepner-like models and Landau–Ginzburg/sigma-model correspondence

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Abstract
The Gepner-like models of k^K-type is considered, when $k + 2$ is a multiple of K the elliptic genus and the Euler characteristic is calculated. Using free-field representation we relate these models to σ-models on hypersurfaces in the total space of anticanonical bundle over the projective space $\mathbb{P}^{K-1}$.

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1. Introduction

Since the famous work of Gepner [1] the geometric aspects underlying his purely algebraic, conformal field theory (CFT) construction of the superstring vacuum have become an area of intensive study. His conjecture that there is some relationship between the CY sigma model and the product of $N = 2$ minimal models has been essentially clarified in the works [1–4, 8–12]. Mirror symmetry, discovered in [2, 5–7], is one of the most important results of this continuing line of research.

In the important work of Borisov [13] the vertex operator algebra endowed with $N = 2$ Virasoro superalgebra action has been constructed for each pair of dual reflexive polytopes defining toric CY manifold. Thus he constructed directly CFT from toric dates of CY manifold. The approach of Borisov is based essentially on the important work of Malikov, Schechtman and Vaintrob [14] where a certain sheaf of vertex algebras, which is called the chiral de Rham complex, has been introduced. Roughly speaking the construction of [14] is a kind of free-field representation known as the 'bc–$\beta\gamma$'-system which in the case of $N = 2$ superconformal sigma model on toric CY is closely related to the Feigin and Semikhatov free-field representation [16] of $N = 2$ supersymmetric minimal models. This circumstance is probably the key to understanding string geometry of Gepner models and proving Gepner's conjecture.

A significant step in this direction has been made in paper [19] where the vertex algebra of a certain Landau–Ginzburg (LG) orbifold has been related to the chiral de Rham complex of toric CY manifold by some spectral sequence. The CY manifold has been realized as an algebraic surface degree K in the projective space $\mathbb{P}^{K-1}$ and one of the key points of [19] is that the free-field representation of the corresponding LG orbifold is given by K copies of $N = 2$ minimal model free-field representation of [16].

In this paper we try to extend the LG/sigma-model correspondence of [19] and consider Gepner-like models which are the products of $N = 2$ minimal models projected by the integer $U(1)$ charge condition. Thus we orbifoldize the product of $N = 2$ minimal models in complete similarity to the case of Gepner models. The only difference is that we relax the total central charge condition for the product of minimal models and consider the product of K-copies of $N = 2$ minimal models with equal central charges $c_1 = \cdots = c_K = \frac{3k}{k+2}$, where $k + 2$ is a multiple of K. When $k + 2 = K$ we are in the CY situation considered in [19]. In the general case we calculate in section 2 the elliptic genus and Euler characteristic of the model. In section 3 we use free-field representation of [16] to relate this model to the $\mathbb{C}^K/\mathbb{Z}_{k+2}$ LG orbifold. In section 4 we discuss briefly the resolution of orbifold singularity and relate the model to the σ-model on a hypersurface in the total space of the anticanonical bundle over the projective space $\mathbb{P}^{K-1}$.

2. The elliptic genus and Euler characteristic of the Gepner-like models

In this section the elliptic genus is calculated for certain orbifold of the product of $N = 2$ minimal models. As a preliminary we represent a collection of known facts on the $N = 2$ minimal models and fix the notation.

2.1. The products of $N = 2$ minimal models

The tensor product of $N = 2$ unitary minimal models taking in a number of K can be characterized by K dimensional vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_K)$, where $\mu_i \geqslant 2$ being integer defines the central charge of the individual model by $c_i = 3\left(1 - \frac{2}{\mu_i}\right)$. For each individual minimal model we denote by $M_{h,t}$ the irreducible unitary $N = 2$ Virasoro superalgebra representation in the NS sector and denote by $\chi_{h,-t}(q, u)$ the character of the representation, where $h = 0, \ldots, \mu - 2$ and $t = 0, \ldots, h$. There are the following important automorphisms of the irreducible modules and characters [16, 17]:

$$M_{h,t} \equiv M_{\mu-h-2,t-h-1}, \qquad \chi_{h,t}(q, u) = \chi_{\mu-h-2,t-h-1}(q, u), \tag{1}$$

$$M_{h,t} \equiv M_{h,t+\mu}, \qquad \chi_{h,t+\mu}(q, u) = \chi_{h,t}(q, u), \tag{2}$$

where μ is odd and

$$M_{h,t} \equiv M_{h,t+\mu}, \qquad \chi_{h,t+\mu}(q, u) = \chi_{h,t}(q, u), \qquad h \neq \left[\frac{\mu}{2}\right] - 1,$$

$$M_{h,t} \equiv M_{h,t+[\frac{\mu}{2}]}, \qquad \chi_{h,t+[\frac{\mu}{2}]}(q, u) = \chi_{h,t}(q, u), \qquad h = \left[\frac{\mu}{2}\right] - 1, \tag{3}$$

where μ is even. In what follows we extend the set of admissible t

$$t = 0, \ldots, \mu - 1 \tag{4}$$

using the automorphisms above.

The parameter $t \in \mathbb{Z}$ labels the spectral flow automorphisms [18] of $N = 2$ Virasoro superalgebra in the NS sector

$$G^{\pm}[r] \to G^{\pm}_t[r] \equiv U^t G^{\pm}[r] U^{-t} \equiv G^{\pm}[r \pm t],$$

$$L[n] \to L_t[n] \equiv U^t L[n] U^{-t} \equiv L[n] + tJ[n] + t^2 \frac{c}{6}\delta_{n,0},$$
$$J[n] \to J_t[n] \equiv U^t J[n] U^{-t} \equiv J[n] + t\frac{c}{3}\delta_{n,0}, \tag{5}$$

where U^t denotes the spectral flow operator generating twisted sectors and r is half-integer for the modes of the spin-3/2 fermionic currents $G^{\pm}(z)$ while n is integer for the modes of stress–energy tensor $T(z)$ and $U(1)$-current $J(z)$ of the $N = 2$ Virasoro superalgebra. So allowing t to be half-integer we recover the irreducible representations and characters in the R sector.

We use the following expression for the characters found in [17]:

$$\chi_{h,-t}(u,q) = q^{\frac{h}{2\mu}+\frac{c}{6}t^2+\frac{th}{\mu}-\frac{c}{24}} q^{\frac{1-\mu}{8}} u^{\frac{h}{\mu}+\frac{ct}{3}} \left(\frac{\eta(q^{\mu})}{\eta(q)}\right)^3$$
$$\times \prod_{n=0} \frac{\left(1+uq^{\frac{1}{2}+t+n}\right)}{\left(1+u^{-1}q^{-\frac{1}{2}-t+n\mu}\right)} \frac{\left(1+u^{-1}q^{\frac{1}{2}-t+n}\right)}{\left(1+uq^{\frac{1}{2}+t+(n+1)\mu}\right)} \frac{\left(1-q^{n+1}\right)}{\left(1-q^{(n+1)\mu}\right)}$$
$$\times \prod_{n=0} \frac{\left(1-q^{-1-h+n\mu}\right)}{\left(1+uq^{-\frac{1}{2}-h+t+n\mu}\right)} \frac{\left(1-q^{1+h+(n+1)\mu}\right)}{\left(1+u^{-1}q^{\frac{1}{2}+h-t+(n+1)\mu}\right)}, \tag{6}$$

where

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1} (1-q^n). \tag{7}$$

The $N = 2$ Virasoro superalgebra generators in the product of minimal models are given by the sums of generators of each minimal model

$$G^{\pm}[r] = \sum_i G_i^{\pm}[r], \qquad J[n] = \sum_i J_i[n], \qquad T[n] = \sum_i T_i[n], \qquad c = \sum_i c_i. \tag{8}$$

This algebra is obviously acting in the tensor products $M_{\mathbf{h},\mathbf{t}} = \otimes_{i=1}^{K} M_{h_i,t_i}$ of the irreducible $N = 2$ Virasoro superalgebra representations of each individual model. We use similar notation for the corresponding product of characters

$$\chi_{\mathbf{h},\mathbf{t}}(q,u) = \prod_{i=1}^{K} \chi_{h_i,t_i}(q,u). \tag{9}$$

By definition [8] the elliptic genus of $N = 2$ supersymmetric CFT is given by

$$\mathrm{Ell}(\tau,\upsilon) = Tr_{(R\times R)}((-1)^{f+\bar{f}} \exp\left[\imath 2\pi\tau\left(L[0]-\frac{c}{24}\right) + \imath 2\pi\upsilon J[0]\right]$$
$$\times \exp\left[\imath 2\pi\bar{\tau}\left(\bar{L}[0]-\frac{c}{24}\right)\right] \tag{10}$$

The trace is taken over the Hilbert space in the $R \times R$ sector and the operators f and $\bar{f}$ are fermion number operators in left-moving and right-moving sectors.

2.2. *Elliptic genus calculation*

Now we calculate the elliptic genus for the case of orbifold of the product of minimal models when the K-dimensional vector is given by $\boldsymbol{\mu} = (\mu, \ldots, \mu)$, where μ is positive and multiple of K. In these models the total central charge is $3K\left(1-\frac{2}{\mu}\right)$, so it is no longer integer and multiple of 3 in general. Except the cases $\mu = K, 2K$ they cannot be considered in general as the models of superstring compactification. Nevertheless, the orbifold projection consistent with modular invariance still exists [3], which makes them interesting $N = 2$ supersymmetric models of CFT from the geometric point of view. The general prescription for the orbifold elliptic genus calculation has been developed in [15] which we shall follow closely.

Before the orbifold projection the elliptic genus of the product of $N = 2$ minimal models can be calculated as the elliptic genus of the LG-model [8, 15]

$$\mathrm{Ell}(\tau, \upsilon) = \prod_{i=1}^{K} \mathrm{Ell}_i(\tau, \upsilon),$$

$$\mathrm{Ell}_i(\tau, \upsilon) = u^{-\frac{c_i}{6}} \frac{\left(1 - u^{1-\frac{1}{\mu}}\right)}{\left(1 - u^{\frac{1}{\mu}}\right)} \prod_{n=1} \frac{\left(1 - u^{1-\frac{1}{\mu}} q^n\right)}{\left(1 - u^{\frac{1}{\mu}} q^n\right)} \frac{\left(1 - u^{-1+\frac{1}{\mu}} q^n\right)}{\left(1 - u^{-\frac{1}{\mu}} q^n\right)}. \tag{11}$$

In fact one can get this expression directly using free-field realization of the $N = 2$ minimal model of section 3 giving thereby the proof of LG-calculation from [8].

The orbifold group is $\mathbb{Z}_\mu$ and generated by

$$g = \exp(\imath 2\pi J[0]). \tag{12}$$

According to [15] the orbifold elliptic genus is given by

$$\mathrm{Ell}_{\mathrm{orb}}(\tau, \upsilon) = \frac{1}{\mu} \sum_{n,l=0}^{\mu-1} \epsilon(n, l) \exp\left(\imath 2\pi \frac{c}{6} nl\right) \prod_{i=1}^{K} \times \exp\left(\imath 2\pi \frac{c_i}{6}(n^2\tau + 2n\upsilon)\right) \mathrm{Ell}_i(\tau, \upsilon + n\tau + l), \tag{13}$$

where

$$\epsilon(n, l) = \exp\left(\imath\pi(n + l + nl)K\right). \tag{14}$$

When $\frac{c}{3} \in \mathbb{Z}$ there is another choice of the coefficients $\epsilon(n, l) : \epsilon(n, l) = \exp\left(i\pi(n + l + nl)\frac{c}{3}\right)$. This case corresponds to the CY manifold realized as a double cover of $\mathbb{P}^{k-1}$ with ramification along some submanifold in $\mathbb{P}^{k-1}$. In this paper we will not consider this case.

The summation over n is due to the spectral flow twisted sector generated by the product of spectral flow twisted operators $\prod_{i=1}^{K} U_i^n$. The summation over l corresponds to the projection on the $\mathbb{Z}_\mu$-invariant states. The Ramond sector is given by the $\frac{1}{2}$-twisted sector. By this convention the chiral-primary fields of the NS sector corresponds to the ground states in the R sector.

The Euler characteristic is given by the value of the elliptic genus at $\upsilon = 0$

$$\mathrm{Eu} \equiv \lim_{\upsilon\to 0} \mathrm{Ell}_{\mathrm{orb}}(\tau, \upsilon) = \frac{(\mu - 1)^K}{\mu} + (-1)^K \frac{\mu^2 - 1}{\mu}. \tag{15}$$

This expression follows from

$$\lim_{\upsilon\to 0} \mathrm{Ell}_i(\tau, \upsilon + n\tau + l) = (-1)^n(\mu_i - 1) \exp\left(-\imath 2\pi \frac{c_i}{6} n^2\tau\right), \quad \text{if} \quad l = 0,$$

$$\lim_{\upsilon\to 0} \mathrm{Ell}_i(\tau, \upsilon + n\tau + l) = (-1)^{n+l+1} \exp\left(\imath 2\pi \frac{nl}{\mu_i}\right) \exp\left(-\imath 2\pi \frac{c_i}{6} n^2\tau\right), \quad \text{if} \quad l > 0. \tag{16}$$

3. LG orbifold geometry of Gepner-like models

In this section we relate the Gepner-like models to the LG orbifolds $\mathbb{C}^K/\mathbb{Z}_\mu$ using essentially the free-field construction of irreducible representations of $N = 2$ minimal models found by Feigin and Semikhatov in [16].

3.1. Free-field realization of $N = 2$ minimal models

Let $X(z)$, $X^*(z)$ be the free bosonic fields and $\psi(z)$, $\psi^*(z)$ be the free fermionic fields (in the left-moving sector) so that its OPEs are given by

$$X^*(z_1)X(z_2) = \ln(z_{12}) + reg., \qquad \psi^*(z_1)\psi(z_2) = z_{12}^{-1} + reg, \tag{17}$$

where $z_{12} = z_1 - z_2$. Then for an arbitrary number μ the currents of $N = 2$ super-Virasoro algebra are given by

$$\begin{aligned}
&G^+(z) = \psi^*(z)\partial X(z) - \frac{1}{\mu}\partial\psi^*(z), \qquad G^-(z) = \psi(z)\partial X^*(z) - \partial\psi(z),\\
&J(z) = \psi^*(z)\psi(z) + \frac{1}{\mu}\partial X^*(z) - \partial X(z),\\
&T(z) = \partial X(z)\partial X^*(z) + \frac{1}{2}(\partial\psi^*(z)\psi(z) - \psi^*(z)\partial\psi(z)) - \frac{1}{2}(\partial^2 X(z) + \frac{1}{\mu}\partial^2 X^*(z)),
\end{aligned} \tag{18}$$

and the central charge is

$$c = 3\left(1 - \frac{2}{\mu}\right). \tag{19}$$

As usual, the fermions are expanded into the half-integer modes in the NS sector and they are expanded into integer modes in the R sector

$$\psi(z) = \sum_r \psi[r]z^{-\frac{1}{2}-r}, \quad \psi^*(z) = \sum_r \psi^*[r]z^{-\frac{1}{2}-r}, \quad G^{\pm}(z) = \sum_r G^{\pm}[r]z^{-\frac{3}{2}-r}. \tag{20}$$

The bosons are expanded in both sectors into the integer modes

$$\begin{aligned}
&\partial X(z) = \sum_{n\in Z} X[n]z^{-1-n}, \qquad \partial X^*(z) = \sum_{n\in Z} X^*[n]z^{-1-n},\\
&J(z) = \sum_{n\in Z} J[n]z^{-1-n}, \qquad T(z) = \sum_{n\in Z} L[n]z^{-2-n}.
\end{aligned} \tag{21}$$

In the NS sector $N = 2$ Virasoro superalgebra is acting naturally in the Fock module F_{p,p^*} generated by the fermionic operators $\psi^*[r], \psi[r], r < \frac{1}{2}$, and bosonic operators $X^*[n], X[n], n < 0$ from the vacuum state $|p, p^*\rangle$ such that

$$\begin{aligned}
&\psi[r]|p,\ p^*\rangle = \psi^*[r]|p,\ p^*\rangle = 0, \qquad r \geqslant \tfrac{1}{2},\\
&X[n]|p,\ p^*\rangle = X^*[n]|p,\ p^*\rangle = 0, \qquad n \geqslant 1,\\
&X[0]|p,\ p^*\rangle = p|p,\ p^*\rangle, \qquad X^*[0]|p,\ p^*\rangle = p^*|p,\ p^*\rangle.
\end{aligned} \tag{22}$$

It is a primary state with respect to the $N = 2$ Virasoro algebra

$$\begin{aligned}
&G^{\pm}[r]|p,\ p^*\rangle = 0, \qquad r > 0,\\
&J[n]|p,\ p^*\rangle = L[n]|p,\ p^*\rangle = 0, \qquad n > 0,\\
&J[0]|p,\ p^*\rangle = \frac{j}{\mu}|p,\ p^*\rangle = 0,\\
&L[0]|p,\ p^*\rangle = \frac{h(h+2) - j^2}{4\mu}|p,\ p^*\rangle = 0,
\end{aligned} \tag{23}$$

where $j = p^* - \mu p, h = p^* + \mu p$.

When $\mu - 2$ is integer and non-negative the Fock modules are highly reducible representations of $N = 2$ Virasoro algebra.

The irreducible module $M_{h,j}$ is given by cohomology of some complex building up from Fock modules. This complex has been constructed in [16]. Let us consider first free-field construction for the chiral module $M_{h,0}$. In this case the complex (which is known due to Feigin and Semikhatov as butterfly resolution) can be represented by the following diagram:

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & \\
\dots & \leftarrow & F_{1,h+\mu} & \leftarrow & F_{0,h+\mu} & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & \\
\dots & \leftarrow & F_{1,h} & \leftarrow & F_{0,h} & & & & & & \\
 & & & & & \nwarrow & & & & & \\
 & & & & & & F_{-1,h-\mu} & \leftarrow & F_{-2,h-\mu} & \leftarrow & \dots \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & F_{-1,h-2\mu} & \leftarrow & F_{-2,h-2\mu} & \leftarrow & \dots \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & \vdots & & \vdots & &
\end{array} \tag{24}$$

The horizontal arrows in this diagram are given by the action of

$$Q^{+} = \oint \mathrm{d}z S^{+}(z), \qquad S^{+}(z) = \psi^{*} \exp(X^{*})(z). \tag{25}$$

The vertical arrows are given by the action of

$$Q^{-} = \oint \mathrm{d}z S^{-}(z), \qquad S^{-}(z) = \psi \exp(\mu X)(z). \tag{26}$$

The diagonal arrow at the middle of butterfly resolution is given by the action of $Q^{+}Q^{-}$. It is a complex due to the following properties of screening charges $Q^{\pm}$:

$$(Q^{+})^{2} = (Q^{-})^{2} = \{Q^{+}, Q^{-}\} = 0. \tag{27}$$

The main statement of [16] is that the complex (24) is exact except at the $F_{0,h}$ module, where the cohomology is given by the chiral module $M_{h,0}$.

To get the resolution for the irreducible module $M_{h,t}$ one can use the observation [16] that all irreducible modules can be obtained from the chiral module $M_{h,0}$, $h = 0, \dots, \mu - 2$ by the spectral flow action U^{-t}, $t = 1, \dots, \mu - 1$. The spectral flow action on the free fields can be easily described if we bosonize fermions ψ^{*}, ψ

$$\psi(z) = \exp(-\phi(z)), \qquad \psi^{*}(z) = \exp(\phi(z)) \tag{28}$$

and introduce the spectral flow vertex operator

$$U^{t}(z) = \exp\left(-t\left(\phi + \frac{1}{\mu}X^{*} - X\right)(z)\right). \tag{29}$$

Using resolution (24) one can get directly expression (11) for the elliptic genus. By the spectral flow we obtain also expression (6) for the character.

The resolutions and irreducible modules in the R sector are generated from the resolutions and modules in the NS sector by the spectral flow operator $U^{\frac{1}{2}}$.

3.2. Free-field realization of the product of minimal models

It is clear how to generalize the free-field representation for the case of tensor product of $N = 2$ minimal models. One has to introduce (in the left-moving sector) the free bosonic fields $X_i(z)$, $X_i^*(z)$ and free fermionic fields $\psi_i(z)$, $\psi_i^*(z)$, $i = 1, \ldots, K$ so that the singular OPEs are given by (17). The $N = 2$ superalgebra Virasoro currents for each of the models are given by (18). To describe the products of irreducible representations $M_{\mathbf{h},\mathbf{t}}$ we introduce the fermionic screening currents and their charges

$$S_i^+(z) = \psi_i^* \exp\left(X_i^*\right)(z), \qquad S_i^-(z) = \psi_i \exp(\mu_i X_i)(z), \qquad Q_i^{\pm} = \oint \mathrm{d}z S_i^{\pm}(z). \tag{30}$$

Then the module $M_{\mathbf{h},0}$ is given by the cohomology of the product of butterfly resolutions (24) for each minimal model. The resolution of the module $M_{\mathbf{h},\mathbf{t}}$ is generated by the spectral flow operator $U^{\mathbf{t}} = \prod_i U_i^{t_i}$, $t_i = 1, \ldots, \mu_i - 1$, where $U_i^{t_i}$ is the spectral flow operator from the ith minimal model (29). Allowing t_i to be half-integer we generate the corresponding objects in the R sector.

3.3. LG orbifold geometry of Gepner-like models

The elliptic genus (13) can be considered as the Euler character of certain complex. It is an orbifold of the complex which is given by the sum of products of butterfly resolutions for the modules $M_{\mathbf{h},0}$. The cohomology of this complex can be calculated by two steps.

At first step we take the cohomology with respect to the operator

$$Q^+ = \sum_{i=1}^{K} Q_i^+. \tag{31}$$

It is generated by the $bc\beta\gamma$ system of fields

$$\begin{aligned} &a_i(z) = \exp[X_i](z), \qquad \alpha_i(z) = \psi_i \exp[X_i](z), \\ &a_i^*(z) = \left(\partial X_i^* - \psi_i\psi_i^*\right)\exp[-X_i](z), \qquad \alpha_i^*(z) = \psi_i^* \exp[-X_i](z). \end{aligned} \tag{32}$$

The fields $a_i(z)$ correspond to the coordinates a_i on the complex space $\mathbb{C}^K$, the fields $a_i^*(z)$ correspond to the operators $\frac{\partial}{\partial a_i}$. The fields $\alpha_i(z)$ correspond to the differentials $\mathrm{d}a_i$, while $\alpha_i^*(z)$ correspond to the conjugated $\mathrm{d}a_i$.

In terms of the fields (32) the $N = 2$ Virasoro superalgebra currents (8) are given by

$$\begin{aligned} &G^- = \sum_i \alpha_i a_i^*, \qquad G^+ = \sum_i \left(1 - \frac{1}{\mu}\right)\alpha_i^*\partial a_i - \frac{1}{\mu}a_i\partial\alpha_i^*, \\ &J = \sum_i \left(1 - \frac{1}{\mu}\right)\alpha_i^*\alpha_i + \frac{1}{\mu}a_i a_i^*, \\ &T = \sum_i \frac{1}{2}\left(\left(1 + \frac{1}{\mu}\right)\partial\alpha_i^*\alpha_i - \left(1 - \frac{1}{\mu}\right)\alpha_i^*\partial\alpha_i\right) + \left(1 - \frac{1}{2\mu}\right)\partial a_i a_i^* - \frac{1}{2\mu}a_i\partial a_i^*. \end{aligned} \tag{33}$$

Note that zero mode $G^-[0]$ is acting on the space of states generated by the $bc\beta\gamma$ system of fields similar to the de Rham differential action on the de Rham complex of $\mathbb{C}^K$. The next important property is the behavior of the $bc\beta\gamma$ system under the change of coordinates on $\mathbb{C}^K$ [14]. It endows the $bc\beta\gamma$ system (32) with the structure of sheaf known as the chiral de Rham complex due to [14]. It provides the geometric meaning to the algebraic $\mathbb{Z}_\mu$-orbifold projection of the product of minimal models.

Indeed, the screening charges Q_i^+ correspond to some cone in the lattice $\mathbb{Z}^K$ generated by the basic vectors e_i. The monomials generated by the fields $a_i(z)$ correspond to the dual cone in the dual lattice [22]. The charges of the fields (32) are given by

$$J(z_1)a_i(z_2) = z_{12}^{-1}\frac{1}{\mu}a_i(z_2) + r., \qquad J(z_1)a_i^*(z_2) = -z_{12}^{-1}\frac{1}{\mu}a_i^*(z_2) + r.,$$
$$J(z_1)\alpha_i(z_2) = -z_{12}^{-1}\left(1-\frac{1}{\mu}\right)\alpha_i(z_2) + r., \qquad J(z_1)\alpha_i^*(z_2) = z_{12}^{-1}\left(1-\frac{1}{\mu}\right)\alpha_i^*(z_2) + r. \tag{34}$$

Hence, making the projection on $\mathbb{Z}_\mu$-invariant states and adding twisted sectors generated by $\prod_{i=1}^{\mu-1}(U_i)^n$ we obtain toric construction of the chiral de Rham complex of the orbifold $\mathbb{C}^K/\mathbb{Z}_\mu$. The chiral de Rham complex on the orbifold has recently been introduced in [21].

The second step in the cohomology calculation is given by the cohomology with respect to the differential $Q^- = \sum_{i=1}^K Q_i^-$. This operator survives the orbifold projection and its expression in terms of fields (32) is

$$Q^- = \oint \mathrm{d}z \sum_{i=1}^K \alpha_i (a_i)^{\mu-1}. \tag{35}$$

Therefore the second step of cohomology calculation gives the restriction of the space of states to the points $\mathrm{d}W = 0$ of the potential

$$W = \sum_{i=1}^K (a_i)^\mu. \tag{36}$$

Thus the total space of states is the space of states of the LG orbifold $\mathbb{C}^K/\mathbb{Z}_\mu$ and expression (13) is the elliptic genus of this LG orbifold.

4. LG/sigma-model correspondence conjecture

As has already been mentioned the case of $\mu = K$ corresponds to the CY manifold which is given by degree K surface in projective space $\mathbb{P}^{K-1}$. The chiral de Rham complex on this manifold has been constructed in [13, 19]. In [19] the chiral de Rham complex on the CY manifold in $\mathbb{P}^{K-1}$ has been calculated by the spectral sequence which relates this complex to the chiral de Rham complex on the LG orbifold.

We briefly consider here the spectral sequence of [19] for the simplest case of 0-dimensional CY manifold in $\mathbb{P}^1$ which corresponds to $\boldsymbol{\mu} = (2, 2)$ model. Then we consider the possible generalization to the case when μ is a multiple of K and discuss the underlying geometry.

When $K = 2$ and $\boldsymbol{\mu} = (2, 2)$ expression (13) gives the elliptic genus of the LG orbifold $\mathbb{C}^2/\mathbb{Z}_2$ with the potential

$$W = a_1^2 + a_2^2 \tag{37}$$

as we have seen in section 3.

According to the construction [13, 19] the resolution of the orbifold singularity is given by the screening charge

$$D_0^+ = \oint \mathrm{d}z \frac{1}{2}(\psi_1^* + \psi_2^*)\exp\left(\frac{1}{2}(X_1^* + X_2^*)\right)(z). \tag{38}$$

It gives a fan [22] consisting of two 2-dimensional cones σ_1 and σ_2, generated in the lattice $\left(\frac{1}{2}\mathbb{Z}\right)^2$ by the vectors $\left(e_1, \frac{1}{2}(e_1+e_2)\right)$ and vectors $\left(e_2, \frac{1}{2}(e_1+e_2)\right)$, correspondingly. To each of the cones σ_i the $bc\beta\gamma$ system of fields is related by the cohomology of the differential $Q_i^+ + D_0^+$ (the first step of cohomology calculation). By the explicit calculations (see, for example, [13]) one can show that these two systems generate the space of sections of the chiral de Rham complex on the open sets of the standard covering of the total space of $O(2)$ line bundle over $\mathbb{P}^1$. The Chech complex of the standard covering glues these sections into the chiral de Rham complex of the total space of the bundle. The cohomology with respect to the differential Q^- restricts the complex to the set of points $\mathrm{d}W = 0$.

Now we propose the orbifold singularity resolution when $K = 2$ and $\mu = 2m, m = 1, 2, \ldots$. In this case we have the LG orbifold $\mathbb{C}^2/\mathbb{Z}_{2m}$ with the potential

$$W = a_1^{2m} + a_2^{2m}. \tag{39}$$

To resolve the orbifold singularity we consider the following set of screening charges:

$$D_n^+ = \oint \mathrm{d}z \left(\frac{m-n}{2m}\psi_1^* + \frac{m+n}{2m}\psi_2^*\right) \exp\left(\frac{m-n}{2m}X_1^* + \frac{m+n}{2m}X_2^*\right)(z),$$
$$n = -m+1, \ldots, m-1. \tag{40}$$

It is easy to check that these operators commute with the total $N = 2$ Virasoro superalgebra currents (33). They commute also with the operators Q_i^- when $\mu = 2m$. But most of the fields (40) cannot appear as marginal operators of the model because they should come from twisted sectors which do not exist in the model. The only exception comes from the spectral flow operator $\prod_{i=1}^{\mu-1}(U_i)^n$. Hence the only screening charge one can add to resolve the singularity is D_0^+, the middle one from (40). By this means we are turning back to the fan of $\boldsymbol{\mu} = (2, 2)$ model. The important difference however is that the group $\mathbb{Z}_m$ is acting on the chiral de Rham complex sections. But the only $bc\beta\gamma$ fields charged with respect to this group correspond to the fibers of the $O(2)$-bundle. In other words, the group $\mathbb{Z}_m$ is acting only along the fibers, so that the base $\mathbb{P}^1$ is the fixed point set of the action. Therefore we obtain after the blow-up the $\mathbb{Z}_m$-orbifold of the chiral de Rham complex of the $O(2)$-bundle total space.

The differential Q^- of the second step cohomology calculation commutes with D_0^+ and survives $\mathbb{Z}_m$-projection. It defines the function (potential) W on the total space of the $O(2)$-bundle and Q^--cohomology calculation restricts the chiral de Rham complex to the $\mathrm{d}W = 0$ point set of the function.

We find the potential by the explicit calculation in some coordinates. According to the construction [13] the set of sections of the chiral de Rham complex of the $O(2)$-bundle over that of the open set Γ_i $(i = 1, 2)$ of the standard covering of the total space of the bundle is given by the cohomology of $Q_i^+ + D_0^+$. For example, the sections of the chiral de Rham complex over the Γ_1 is given by $Q_1^+ + D_0^+$ cohomology and generated by the following $bc\beta\gamma$ fields:

$$b_0(z) = \exp[2X_2](z), \qquad \beta_0(z) = 2\psi_2 \exp[2X_2](z),$$
$$b_0^*(z) = \left(\tfrac{1}{2}\left(\partial X_1^* + \partial X_2^*\right) - 2\psi_2 \tfrac{1}{2}(\psi_1^* + \psi_2^*)\right) \exp[-2X_2](z),$$
$$\beta_0^*(z) = \tfrac{1}{2}\left(\psi_1^* + \psi_2^*\right) \exp[-2X_2](z), \tag{41}$$
$$b_1(z) = \exp[X_1 - X_2](z), \qquad \beta_1(z) = (\psi_1 - \psi_2)\exp[X_1 - X_2](z),$$
$$b_1^*(z) = \left(\partial X_1^* - (\psi_1 - \psi_2)\psi_1^*\right)\exp[-X_1 + X_2](z), \qquad \beta_1^*(z) = \psi_1^* \exp[-X_1 + X_2](z).$$

Then the potential (37) takes the form

$$W = (b_0)^m(1 + (b_1)^{2m}). \tag{42}$$

The $\mathrm{d}W = 0$ points are given by the equations

$$(b_0)^{m-1} = 0, \quad \text{when} \quad b_1^{2m} \neq -1, \qquad (b_0)^m = 0, \quad \text{when} \quad b_1^{2m} = -1. \tag{43}$$

Analogously, the sections of the chiral de Rham complex over the Γ_2 are given by the $Q_2^+ + D_0^+$ cohomology and generated by the fields

$$\begin{aligned}
&\tilde{b}_0(z) = \exp[2X_1](z), \qquad \tilde{\beta}_0(z) = 2\psi_1 \exp[2X_1](z),\\
&\tilde{b}_0^*(z) = \left(\tfrac{1}{2}\left(\partial X_1^* + \partial X_2^*\right) - 2\psi_1 \tfrac{1}{2}\left(\psi_1^* + \psi_2^*\right)\right)\exp[-2X_1](z),\\
&\tilde{\beta}_0^*(z) = \tfrac{1}{2}\left(\psi_1^* + \psi_2^*\right)\exp[-2X_1](z),\\
&\tilde{b}_1(z) = \exp[-X_1 + X_2](z), \qquad \tilde{\beta}_1(z) = -(\psi_1 - \psi_2)\exp[-X_1 + X_2](z),\\
&\tilde{b}_1^*(z) = \left(\partial X_2^* - (-\psi_1 + \psi_2)\psi_2^*\right)\exp[X_1 - X_2](z), \tilde{\beta}_1^*(z) = \psi_2^* \exp[X_1 - X_2](z).
\end{aligned} \tag{44}$$

In these coordinates the potential takes the form

$$W = (\tilde{b}_0)^m (1 + (\tilde{b}_1)^{2m}) \tag{45}$$

so that $\mathrm{d}W = 0$ points set is given similar to (43).

Comparing expressions (41) and (44) we see that field $b_0(z)$ ($\tilde{b}_0(z)$), corresponds to the coordinate along the fiber and the field $b_1(z)$ ($\tilde{b}_1(z)$) corresponds to the coordinate along the base $\mathbb{P}^1$ of $O(2)$-bundle in the open set Γ_1 (Γ_2).

For general values of K and $\mu = mK$ the situation is similar. The only screening charge one can add to resolve the orbifold singularity comes from the spectral flow operator

$$D_0^+ = \oint \mathrm{d}z \frac{1}{K}\left(\sum_i \psi_i^*\right)\exp\left(\frac{1}{K}\sum_i X_i^*\right)(z). \tag{46}$$

Together with Q_i^+ it gives the standard fan of the $O(K)$-bundle total space over $\mathbb{P}^{K-1}$. The highest dimensional cones σ_i of the fan are labeled by the sets $\left(D_0^+, Q_1^+, \ldots, Q_{i-1}^+, Q_{i+1}^+, \ldots, Q_K^+\right)$. The group $\mathbb{Z}_m$ is acting along the fibers of the bundle with the fixed point set $\mathbb{P}^{K-1}$. Thus we obtain after the blow-up the $\mathbb{Z}_m$-orbifold of the chiral de Rham complex of the $O(K)$-bundle total space. The differential Q^- commutes with D_0^+ due to the condition $\mu = Km$ and survives $\mathbb{Z}_m$-projection hence, it defines the potential (36) on the total space of the $O(K)$-bundle. Therefore expression (13) is the elliptic genus of the orbifold of the $O(K)$-bundle restricted to the set of points $\mathrm{d}W = 0$

$$(b_0)^{m-1} = 0, \quad \text{when} \sum_{i=1}^{K-1} b_i^{Km} \neq -1, \qquad (b_0)^m = 0, \quad \text{when} \sum_{i=1}^{K-1} b_i^{Km} = -1, \tag{47}$$

where b_0 is the coordinate along the fiber and b_i, $i = 1, \ldots, K-1$ are the coordinates along the base $\mathbb{P}^{K-1}$ in some of the open set of the standard covering of the $O(K)$-bundle. The algebraic manifold determined by equations (47) is singular except the case $m = 1$. Nevertheless, the Euler characteristics (15) can be represented in the form compatible with these equations

$$\begin{aligned}
\mathrm{Eu} &= (-1)^K \left(K + \frac{(1 - mK)^K - 1}{mK} + (m-1)K\right)\\
&= (-1)^K (m\,\mathrm{Eu}(V) + (m-1)\,\mathrm{Eu}(\mathbb{P}^{K-1} \setminus V)),
\end{aligned} \tag{48}$$

where V is the set of points in $\mathbb{P}^{K-1}$ satisfying the equation $\sum_{i=1}^{K-1} b_i^{Km} + 1 = 0$.

The important peculiarity of the orbifold projection is the coefficients (14). They determine the action of $\mathbb{Z}_m$-group in the twisted and untwisted sectors and govern in particular the topological properties of the σ-model. The investigation of this point as well as more detailed investigation of toric geometry of the models is left for the future.

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